

Finite Size Scaling and Conformal Curves

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February 1, 2008

Abstract

In this letter we investigate the finite size scaling effect on $SLE(\kappa, \rho)$ and boundary conformal field theories and find the effect of fixing some boundary conditions on the free energy per length of $SLE(\kappa, \rho)$. As an application, we will derive the entanglement entropy of quantum systems in critical regime in presence of boundary operators.

Keywords: Conformal Field Theory, Stochastic Loewner Evolution (SLE), Finite Size Scaling

1 introduction

Conformal field theories have found many applications in classification of phase transitions and critical phenomena in two dimensions. In particular the minimal models introduced in [1] reveal many exact solutions to various two dimensional phase transitions like Ising model at critical point or Pott's model and so on [2]. These models were first considered on the whole plane, but as many surface phenomena are very interesting to analyze, boundary conformal field theory was soon developed [3]. Essentially it was shown that conformal field theory in the half plane with proper boundary condition could be mapped to a whole-plane conformal field theory with just holomorphic part.

On the other hand, recently a new method to investigate the so called geometrical phase transitions has been developed, which were previously described by conformal field theories. The new method, Stochastic Loewner Evolution (SLE) [4] is a probabilistic approach to study scaling behavior of geometrical models. SLE's, which are characterized by a parameter κ , can be simply stated as conformally covariant processes, defined on the upper half plane, which describe the evolution of random domains, called SLE hulls. These random domains represent critical clusters. The idea of SLE was first developed by Schramm [4]. He showed that under assumption of conformal invariance, the scaling limit of loop erased random walks is SLE_2 . Since such problems can be investigated either by using SLE methods or by exploiting conformal field theories, there should be a direct relation between the two, such relation was found in [5] and later in [6].

More recently a generalization of these theories has emerged: the so called $SLE(\kappa, \rho)$ [7, 8]. The new theories describe random growing interfaces in a planar domain which have

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markovian property and conformal invariancy. In fact $\text{SLE}(\kappa, \rho)$ is the minimal way to generalize the original SLE while keeping self-similarity and markov property. These models are related to conformal field theory, too [9, 10].

The problem which has not been addressed yet, is the effect of finite size scaling on $\text{SLE}(\kappa, \rho)$ which is in fact the problem of finite size effect on boundary conformal field theory. The problem could appear if you are investigating a system in which the size of the SLE hulls are comparable with the size of the whole system, and this happens in most of physical cases. Also it could be used to calculate the quantum entanglement entropy for quantum systems in $1 + 1$ dimensions, which is a very interesting problem in quantum computation.

In the next section we will briefly recall the properties of $\text{SLE}(\kappa, \rho)$ and its relation to conformal field theory, and in the third section we'll derive the finite size effects and its in different problems.

2 $\text{SLE}(\kappa, \rho)$ and its properties

As mentioned above, $\text{SLE}(\kappa, \rho)$ describes random growing interfaces in a planar domain which have markovian property and conformal invariancy. Schramm-Loewner equation reveals the evolution of conformal map which transforms the remaining part of the upper half plane to the whole upper half plane. The modified version of the equation depends on a series of numbers ρ_j in addition to the points x_j . In the CFT picture, at these point the proper boundary changing operators are inserted [9]. In fact using the $\text{SLE}(\kappa, \rho)$ equation, one is able to show that there should be time independent states $|h, x_j\rangle$, which should annihilate if a certain combination of generators of Virasoro algebra is applied on them. It is then shown in [9] that the operators associated with jumps in piecewise Dirichlet boundary condition of a coulomb gas theory can create these states. In [11] we have generalize the argument to the other boundary conditions in coulomb gas model. Also the effect of existence of a charge at infinity is discussed. To be more specific we will briefly recall the method used in [9, 11]

Considering the coulomb gas action $S = (4\pi)^{-1} \int [(\partial\varphi)^2 + 2Q R\varphi]$, one can derive the classical solution of equation of motion which satisfies the boundary condition of the problem. As an example, in Neuman boundary condition (NBC) the solution has the form $\varphi_{cl} = \sum \lambda_j \ln(z - x_j)(\bar{z} - x_j)$ where λ_j are constants coming from the specific choice of the boundary condition. Then one can consider fluctuations around this solution and find the partition function associated with φ_{cl} which we call it Z_{cl} . This partition function is considered to be the definition of boundary changing operators via defining their correlation functions: $\langle \prod_j \phi_{\lambda_j}(x_j) \rangle = Z_{cl}$. Also any expectation value $\langle O \rangle$ in presence of this boundary condition can be defined in the following way:

$$\langle O \rangle = \frac{\langle O \prod_j \phi_{\lambda_j}(x_j) \rangle}{\langle \prod_j \phi_{\lambda_j}(x_j) \rangle}. \quad (1)$$

In particular the expectation value of energy-momentum tensor T can be computed to be

$$\langle T \rangle = -(\partial\varphi_{cl})^2 + 2Q\partial^2\varphi_{cl} = \sum_{j,k} \frac{\lambda_j \lambda_k}{(z - x_j)(z - x_k)} + 2Q \sum_j \frac{\lambda_j}{(z - x_j)^2}. \quad (2)$$

OPE of this energy momentum tensors with the boundary fields φ_λ could be calculated to derive the weight of these fields $h_\lambda = \lambda(\lambda + 2Q)$.

Using the above equations, the effects of Virasoro generators on the boundary operators could be calculate and one observes that they satisfy the relation coming from SLE part of the

problem if we impose the condition $Q = \frac{\lambda(-\kappa + 4)}{4}$. So the relation between $\text{SLE}(\kappa, \rho)$ and boundary CFT is derived, $\text{SLE}(\kappa, \rho)$ martingales are on one side and boundary conformal field theory is on the other. With this equivalency one can use CFT methods to compute the effect of finite size scaling on $\text{SLE}(\kappa, \rho)$.

3 Finite Size Scaling

To investigate the finite size scaling effect on $\text{SLE}(\kappa, \rho)$, we consider a generic conformal field theory living on the half complex plane. Applying the transformation $z \rightarrow w = \frac{L}{\pi} \ln z$, the half plane is mapped to a strip of length L . Here w and z are the holomorphic coordinates on the strip and the upper half plane, respectively. The energy-momentum tensor on the strip is found to be

$$T_{\text{strip}}(w) = \left(\frac{\pi}{L}\right)^2 \left(T_H(z)z^2 - \frac{c}{24}\right), \quad (3)$$

where T_H is the energy-momentum tensor of the half plane and $c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}$ is the central charge of the theory and is related to a soft breaking of conformal symmetry by introducing a microscopic scale in the system. In our case, the expectation value of energy-momentum tensor is given by equation (2), so in the strip geometry, the vacuum energy density is found to be

$$\langle T_{\text{strip}}(w) \rangle = \left(\frac{\pi}{L}\right)^2 \sum_j \sum_k \frac{\lambda_j(\lambda_k + 2Q\delta_{jk})z^2}{(z - x_j)(z - x_k)} - \left(\frac{\pi}{L}\right)^2 \frac{c}{24}. \quad (4)$$

With this vacuum energy density, the Helmholtz free energy, F , could be computed to see how it changes if the length scale L is varied. This is a straightforward calculation [2] and one finds that if a small variation $\delta L = \epsilon L$ is applied to the strip width, the free energy varies in the following way

$$\delta F = \int \left(\frac{-\pi c}{24L^2} - \frac{\pi}{L^2} \sum_{j,k} \lambda_j(\lambda_k + 2Q\delta_{jk}) \right) dw^1 \delta L. \quad (5)$$

Integrating over L , the free energy of a $\text{SLE}(\kappa, \rho)$ theory per unit length of the strip, F_u , is derived,

$$F_u = \frac{-\pi c}{24L} + \frac{-\pi}{L} \sum_{j,k} \lambda_j(\lambda_k + 2Q\delta_{jk}). \quad (6)$$

The first term of the above equation is the usual part of free energy in strip geometry, while the second term comes from the presence of special boundary condition and reveals the specific finite size properties of $\text{SLE}(\kappa, \rho)$. In the case where $\sum \lambda_k + 2Q = 0$, boundary conditions do not change the free energy. If the boundary changing operators be vertex operators of coulomb gas, this condition which is the neutrality condition, should be satisfied. That is, in the case which the boundary operators are vertex operators, the free energy per length is independent of boundary condition.

The whole procedure could be seen from a different point of view, to derive central charge of some theories. Consider that the charge Q was not inserted in the coulomb gas action, so the central charge would be one. Redoing all the above procedure one arrives at the equation (6) with the charge term being absent. Then the left hand side of this equation could be read as $-\pi c'/24L$ with c' being the effective central charge which is read to be $c' = 1 + 24(\sum \lambda_j)^2$. Thus if the sum of all the charges in the theory be $2Q = \sum \lambda_j$, then the new central charge would be $c' = 1 + 96Q^2$. This argument is not restricted to vertex operators and could be

applied to any primary field. If we use the ward identity of scale invariancy the result is similar to the above: the change in free energy is zero if all primary operators are in the finite distance, but sending one of the operators to infinity, one finds that the difference in the free energy equal to $\frac{\pi h \lambda}{L}$. The simplest case is the ordinary SLE with one operator at origin and the other at infinity. The change in free energy can be computed readily, since both of the operators have equal weight $h = \frac{6-\kappa}{2\kappa}$. A more complex case is SLE on polygon [13], in this case we have $\sum \lambda_j = \frac{\sqrt{\kappa}}{2}$ and the free energy will be $\frac{-\pi c}{24L} - \frac{-\pi\sqrt{\kappa}}{2L}(\frac{\sqrt{\kappa}}{2} + 2Q)$. This argument can also be applied to nSLE's [12], in this case $\rho_i = 2$ and so $\sum \lambda_i = \frac{n}{\sqrt{\kappa}}$ and the free energy can be computed easily.

The equation (6) has some other important results in other area, like quantum information theory. There is a relationship between the free energy of a two dimensional classical statistical system on a strip of infinite length and finite width L and the ground state energy of a one dimensional quantum system as temperature $T \sim \frac{1}{\kappa L}$, so instead of considering the finite temperature quantum theory, one can use the equation (6) to find the entropy of the corresponding quantum system and then the specific heat of the model.

Additionally the method we have used to derive (6) is applicable to quantum entanglement entropy. In the second method of [14] to find the quantum entanglement entropy of a one dimensional quantum system on critical regime, one uses the expectation value of energy-momentum tensor. Now we are able to find the same quantity in presence of some additional boundary operators. If the boundary operators be primary ones, then the resulting entropy will not change. This comes directly from Ward identities associated with scale invariancy, and so it's true for arbitrary number of primary operators and also for more complex CFTs like logarithmic CFTs. In these models we should use different operator product expansions for the logarithmic operators. Again sending the operators to infinity then the free energy changes similar to equation (6) both for ordinary CFT and LCFT.

The last point is that we are also able to find the behavior of largest eigenvalue of the corresponding quantum system $\ln \lambda_0 = \frac{\pi c}{24L} + \frac{\pi}{L} \sum \lambda_j \lambda_k + 2Q \sum \lambda_j$ as L tends to infinity. Note that this is true if we are dealing with a unitary CFT, as in nonunitary models some correlators may grow with distance.

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